Continuous Homomorphisms as Arithmetical Functions, and Sets of Uniqueness

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This is a survey paper on the characterization of continuous group homomorphisms as arithmetical functions, and on sets of uniqueness with respect to completely additive functions.

1 Introduction

Let, as usual $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ be the set of positive integers, integers, rational, real, and complex numbers, respectively. Let $\mathbb{Q}_\times, \mathbb{R}_\times$ be the multiplicative group of positive rationals, reals, respectively. Let $\mathcal{P}$ be the set of prime numbers.

For an arbitrary, additively written Abelian group G let $\mathcal{A}_G$, resp. $\mathcal{A}_G^*$ denote the classes of additive, resp. completely additive functions. A function $f : \mathbb{N} \to G$ belongs to $\mathcal{A}_G$ if $f(nm) = f(m) + f(n)$ holds for each pair of coprime m, n, and it belongs to $\mathcal{A}_G^*$ if the above equation holds for all pairs $m, n \in \mathbb{N}$. If G is written multiplicatively, then we shall write $\mathcal{M}_G, \mathcal{M}_G^*$ instead of $\mathcal{A}_G, \mathcal{A}_G^*$, and the corresponding functions are called multiplicative, completely multiplicative.

If $G = \mathbb{R}$, then we shall write simply $\mathcal{A}, \mathcal{A}^*$ instead of $\mathcal{A}_\mathbb{R}, \mathcal{A}_\mathbb{R}^*$.

If $f \in \mathcal{A}_G^*$, then its domain $\mathbb{N}$ can be extended to $\mathbb{Q}_\times$ by

$$f\left(\frac{m}{n}\right) := f(m) - f(n),$$

and the functional equation

$$f(r_1 r_2) = f(r_1) + f(r_2)$$

remains valid for every $r_1, r_2 \in \mathbb{Q}_\times$.

Let us assume that G is a topological group and $f : \mathbb{Q}_\times \to G$ is continuous at 1. Then for each $\alpha \in \mathbb{R}_\times$ there exists the limit

$$\lim_{r \to \alpha} f(r) =: \Phi(\alpha),$$

Φ is continuous everywhere in $\mathbb{R}_\times$, furthermore $\Phi(\alpha\beta) = \Phi(\alpha) + \Phi(\beta)$ valid for all $\alpha, \beta \in \mathbb{R}_\times$, i.e. Φ is a continuous homomorphism of $\mathbb{R}_\times$ into G.

On the other hand, if $\Phi : \mathbb{R}_\times \to G$ is a homomorphism, then its restriction to the domain $\mathbb{N}$ is a completely additive function.

Let S be an R-module, containing at least two elements, defined over an integral domain R which has an identity. Consider the set of all doubly infinite sequences $(\dots s_{-1}, s_0, s_1, \dots)$ of elements of S. We introduce the shift operator E whose action takes a typical sequence $\{s_n\}$ to the new sequence $\{s_{n+1}\}$. If

$$P(x) = \sum_{j=0}^{r} c_j x^j$$

is a polynomial with coefficients in R, we extend this definition by defining

$$P(E)s_n = \sum_{j=0}^{r} c_j s_{n+j}.$$

In this way we define a ring of operators which is isomorphic to the ring of polynomials with coefficients in R. Let I be the identity operator, and $\Delta := E - I$.

We shall say that an additive function f is of finite support, if it vanishes on the set of prime powers except possibly on the powers of finitely many primes.

For $z \in \mathbb{R}$ let $\|z\| := \min_{k\in\mathbf{Z}} |z - k|$.

2 Characterization of log as an Additive Arithmetical Function

The function $f(n) = \log n$ belongs to $\mathcal{A}^*$. Normally log is considered as a mapping $\mathbb{R}_\times \to \mathbb{R}$ and in this context it is wellknown that continuity along with the functional equation $f(xy) = f(x) + f(y)$ characterizes the logarithm up to a constant factor. Restricting the domain from $\mathbb{R}_\times$ to $\mathbb{N}$ creates an interesting question: What further properties along with the (complete) additivity will ensure that an arithmetic function f is in fact $c \log n$.

The first result of this type was proved by P. Erdös [1] in 1946.

Theorem 1 *If $f \in \mathcal{A}$ and $\Delta f(n) \geq 0$ for all n, or $f(n) \to \infty$ $(n \to \infty)$, then $f(n)$ is a constant multiple of* $\log n$.

In [2] we proved

Theorem 2 *If $f \in \mathcal{A}$ and* $\lim \inf \Delta^k f(n) \geq 0$ *with some $k \in \mathbb{N}$, then f is a constant multiple of* $\log n$.

An important progress has been achieved by E. Wirsing [3] proving the following conjecture of Erdös.

Theorem 3 *If $f \in \mathcal{A}$ and $\Delta f(n) \geq -K$ with some constant K, then $f(n) = c \log n + u(n)$, where $u(n)$ is bounded and c is a suitable constant.*

Another one of Erdös's conjecture was proved in [4].

Theorem 4 *If $f \in \mathcal{A}$ and*

$$\frac{1}{x}\sum_{n\leq x} |\Delta f(n)| \longrightarrow 0, \tag{2.1}$$

then $f = c$ log.

Somewhat later the condition (2.1) was weakened by E. Wirsing. Namely, he proved in [5]

Theorem 5 *Let $f \in \mathcal{A}$. Assume that there exists a constant $\gamma > 1$ and a sequence $x_1 < x_2 < \dots$ such that*

$$x_i^{-1} \sum_{x_i < n \le \gamma x_i} |\Delta f(n)| \longrightarrow 0 \quad (i \longrightarrow \infty).$$

Then $f = c \log$.

By making use of very original new ideas and some deep results on the distribution of primes in arithmetical progressions, E. Wirsing [6] proved

Theorem 6 *If $f \in \mathcal{A}^*$ and $\Delta f(n) = o(\log n)$, then $f(n) = c \log n$.*

One can show easily that the following generalization of the preceding theorems hold true.

Theorem 7

(1) *Let $f, g \in \mathcal{A}$. If*

  (a) *$g(n+1) - f(n) \to 0$, then $f = g = c \log$;*

  (b) *$g(n+1) - f(n)$ is bounded, then $f(n) = c \log n + u(n)$, $g(n) = c \log n + v(n)$, and u, v are bounded.*

(2) *Let $f, g \in \mathcal{A}^*$. If $g(n+1) - f(n) = o(\log n)$, then $f(n) = g(n) = c \log n$.*

For the method of the proof of Theorem 7 see [7], [8].

In [9] and [10] I asked for a characterization of those additive functions which satisfy

$$f(an+b) - f(An+B) \longrightarrow C \quad \text{as} \quad n \longrightarrow \infty \tag{2.2}$$

for some integers $a > 0$, $A > 0$, b, B, and real constant C. I considered it with $B = 0$ and small values of a and b in [9] and [10].

With general a and b but still with $B = 0$ satisfactory results has been achieved by Mauclaire [11].

Elliott solved this problem completely. Namely he demonstrated in [12] that if (2.2) holds, then there is a constant F such that

$$f(m) = F \log m$$

holds for all m coprime to $aA\Delta$, where $\Delta = aB - Ab$, assuming $\Delta \ne 0$. Moreover he could give the values of f for those prime powers p^α for which $p | aA\Delta$.

Another important assertion proved by Elliott is formulated as

Theorem 8 *Assume that $aA\Delta \ne 0$. There exist positive constants c, c_1 so that*

$$\left| \frac{f(m)}{\log m} - \frac{f(n)}{\log n} \right| \le c_1 \left(\frac{L(m)}{\log m} + \frac{L(n)}{\log n} \right)$$

holds uniformly for all integers m and n which satisfy $2 \le m \le n \le e^m$ *and are prime to* $aA\Delta$. *Here*

$$L(x) = \max_{n \le x^c} |f(an+b) - f(An+B)|.$$

The constants c, c_1 *may depend on* a, b, A, B.

The best source for the proof of this theorem and other important results is the excellent book of Elliott [13]. Theorem 8 generalizes a result of Wirsing [6] which sounds as follows:

Let $\beta(x)$ be a positive non-decreasing function so that $\beta(x^2) \le 2^{6/5}\beta(x)$. Let $f \in \mathcal{A}$ such that $f(2) \ge 0$ and $f(n+1) - f(n) \le \beta(n)$, for every $n \in \mathbb{N}$. Then, there is a suitable constant γ so that

$$\left|\frac{f(m)}{\log m} - \frac{f(n)}{\log n}\right| \le \gamma \left(\frac{\beta(m)}{\log m} + \frac{\beta(n)}{\log n}\right)$$

uniformly for $2 \le m \le n \le e^m$.

We shall say that a sequence of real numbers $t_n (n \in \mathbb{N})$ is tight if

$$\limsup_{x \to \infty} \frac{1}{x} \#\{n \le x, |t_n| > K\} =: c(K) \longrightarrow 0 \quad \text{as} \quad K \longrightarrow \infty. \tag{2.3}$$

A. Hildebrand [14] proved that $f(n+1) - f(n)$, $f \in \mathcal{A}$ has a limit distribution if and only if there exists a constant c such that $h(n) := f(n) - c \log n$ satisfies

$$\sum_p \frac{\min(1, h^2(p))}{p} < \infty. \tag{2.4}$$

Though explicitly it was not formulated but from this argument the following assertion follows immediately

Theorem 9 *Let* $f \in \mathcal{A}$. *Then* (2.3) *holds for* $t_n = f(n+1) - f(n)$, *if and only if* (2.4) *is satisfied.*

Later Elliott [15] went on to prove the following more general

Theorem 10 *Let* $a > 0$, $A > 0$, b, B *be integers which satisfy* $aB \ne Ab$, *and* $\eta(x)$ *a real-valued function defined for* $x \ge 2$. *Let* $f_1, f_2 \in \mathcal{A}$, *and* $\eta(x)$ *be an arbitrary function. Let*

$$F_x(z) := \frac{1}{x} \#\{n \le x \mid f_1(an+b) - f_2(An+B) - \eta(x) \le z\}.$$

The following three propositions are equivalent.

(1) *There is an* $\eta(x)$ *so that the frequencies* $F_x(z)$ *converge weakly to a distribution function as* $x \to \infty$.
(2) *There is an* $\eta(x)$ *so that*

$$\lim_{z \to \infty} \limsup_{x \to \infty} (1 - F_x(z) + F_x(-z)) = 0.$$

(3) *There are real numbers* c_1, c_2 *such that for* $h_j(n) := f_j(n) - c_j \log n$ *the conditions*

$$\sum_{p \in \mathcal{P}} \frac{\min(1, h_j^2(p))}{p} < \infty$$

hold.

Let $\varphi : [0, \infty) \to [0, \infty)$ be a so-called subadditive function, i.e. monotonically increasing, $\varphi(x) \to \infty$ as $x \to \infty$, and the condition

$$\varphi(x + y) \le c_1(\varphi(x) + \varphi(y)) \quad \text{for} \quad x, y \ge 1 \tag{2.5}$$

holds with a suitable constant $c_1 > 0$.

We are interested in giving necessary and sufficient conditions for an additive f to satisfy

$$\sum_{n \le x} \varphi(|f(n+1) - f(n)|) \ll x \quad (x \longrightarrow \infty) \tag{2.6}$$

Applying the argument we used in our paper [16] written jointly with Indlekofer, one gets

Theorem 11 *Let* φ *be a subadditive function. The relation* (2.6) *holds for an additive function* f *if and only if there exists a constant* c *such that* $h(n) := f(n) - c \log n$ *satisfies* (2.4) *and*

$$\sum_{|h(q^m)| \ge 1} \frac{\varphi(|h(q^m)|)}{q^m} < \infty, \tag{2.7}$$

where q^m *runs over the set of prime powers.*

Proof: Necessity. Assume that (2.6) holds. Then $\Delta f(n)$ is a tight sequence, and so, by Theorem 8 we obtain the fulfilment of (2.4). Since $\Delta f(n) = \Delta h(n) + o(1)$, therefore $\sum_{n \le x} \varphi(|\Delta h(n)|) \ll x$. Let $h(n)$ be written as the sum of the additive functions $h_1(n), h_2(n)$, where h_1 is a strongly additive function defined for primes q such that

$$h_1(q) = \begin{cases} h(q) & \text{if } |h(q)| < 1, \text{ or if } q = 2 \\ 0 & \text{otherwise,} \end{cases}$$

and $h_2(n)$ is defined by $h_2(n) := h(n) - h_1(n)$.

From (2.5) one gets easily that $\varphi(x) \ll x^c$ for $x \ge 1$ with a suitable constant c. Furthermore, from the generalized Turán-Kubilius inequality due to Elliott (see Lemma 1.4.[13]), together with (2.4) we obtain that $\sum_{n \le x} \varphi(|\Delta h_1(n)|) \ll \sum_{n \le x} |\Delta h_1(n)|^c \ll x$, consequently, from the assumptions (2.6), (2.5), and $|\Delta h_2(n)| \le |\Delta h_1(n)| + |\Delta h(n)|$ we obtain that

$$\sum_{n \le x} \varphi(|\Delta h_2(n)|) \le c_3 x. \tag{2.8}$$

with a suitable constant c_3, for all $x \ge 2$. From (2.8) we obtain (2.7) readily. Let $\tilde{\mathcal{P}}$ be the set of those primes q for which $|h(q)| \ge 1$. As we know (see (2.4)) $\sum_{q \in \tilde{\mathcal{P}}} \frac{1}{q} < \infty$. Let us choose an arbitrary $Y \ge 1$. For all β, $(2 <)2^\beta \le Y$ consider those integers $n = 2^\beta \gamma$, γ odd for which $\gamma(n+1)$ is square-free and coprime to $\tilde{\mathcal{P}}$. By making use of the Eratosthenian sieve we can see that the density of these integers is $\frac{e}{2^\beta}$ with a positive constant e which may depend only on $\tilde{\mathcal{P}}$. Since $h_2(2^\beta\gamma) = h_2(2^\beta)$, and the sequences defined for different β are disjoint ones, from (2.8) we get that

$$\sum_{2^\beta \le Y} \varphi(|h_2(2^\beta)|)\frac{e}{2^\beta} < c_3. \tag{2.9}$$

Let now $\mathcal{B} = \{q^m, q > 2, m \ge 2\} \cup \{q \in \tilde{\mathcal{P}}, q \ne 2\}$. Let $Q \le Y$, $\mathcal{S}_Q$ be the set of those integers $n = 2Q\nu$ for which ν is odd, and $\nu(2Q\nu+1)$ is square free and coprime to $\tilde{\mathcal{P}}$. By the Eratosthenian sieve we obtain that the asymptotic density of $\mathcal{S}_Q$ is $\ge e_1/Q$ with a positive constant e_1. Since $h_2(2Q\nu) = h_2(2) + h_2(Q)$, and the sets $\mathcal{S}_Q$ are disjoint, we obtain that

$$\sum_{Q<Y} \varphi(|h_2(2) + h_2(Q)|)\frac{e_1}{Q} < c_3.$$

Hence we get (2.7) immediately.

Sufficiency. Assume that (2.4), (2.7) hold true. Since $\varphi(|\Delta f(n)|) \ll \varphi(|c\Delta \log\ n|) + \varphi(|\Delta h_1(n)|) + \varphi(|\Delta h_2(n)|)$, therefore summing over n up to x, the first two sums on the right hand side are bounded by x, it remains to prove that

$$\sum_{n\le x} \varphi(|\Delta h_2(n)|) \ll x,$$

which will follow if we show that

$$\sum_{n\le x} \varphi(|h_2(n)|) \ll x. \tag{2.10}$$

Let $\mathcal{T}$ denote the set of those integers D for which $p\|D$ implies that $h_2(p) \ne 0$. The left hand side of (2.10) is bounded by

$$x \sum_{\substack{D\in\mathcal{T}\\ D\le x}} \frac{\varphi(|h_2(D)|)}{D}. \tag{2.11}$$

Iterating (2.5) we obtain that

$$\varphi(|h_2(D)|) \le \sum_{q^m\|D} c^{\omega(\frac{D}{q^m})}\varphi(|h_2(q^m)|),$$

where $\omega(n)$ is the number of distinct prime divisors of n. Thus we have

$$\sum_{\substack{D\in\mathcal{T}\\ D\le x}} \frac{\varphi(|h_2(D)|)}{D} \le 1 + \sum_{D}\sum_{q^m\|D} \frac{\varphi(|h_2(q^m)|)}{q^m}\frac{c^{\omega(\frac{D}{q^m})}}{\frac{D}{q^m}}$$

$$\le 1 + \left(\sum_{q^m\in\mathcal{T}} \frac{\varphi(|h_2(q^m)|)}{q^m}\right)\left\{\sum_{D_1\in\mathcal{T}} \frac{c^{\omega(D_1)}}{D_1}\right\}.$$

On the right hand side both sums are convergent, and the proof is complete. □

As a special case we have the following

Corollary 1 *Let $f \in \mathcal{A}$. The inequality*

$$\sum_{n \le x} |\Delta f(n)|^{\alpha} \ll x$$

holds with some constant $\alpha > 0$, if and only if there is a suitable constant c such that for $h(n) := f(n) - c \log n$

$$\sum_{|h(p)|<1} \frac{h^2(p)}{p} < \infty,$$

and

$$\sum_{|h(q^m)| \ge 1} \frac{|h(q^m)|^{\alpha}}{q^m} < \infty$$

hold.

This assertion for $\alpha = 2$ was proved earlier by Elliott [17].

3 Characterization of n^s as a Multiplicative Function

In a series of papers ([18] I–VI) I considered functions $f \in \mathcal{M}$ under the conditions that $\Delta f(n)$ tends to zero in some sense. I could determine all those functions $f, g \in \mathcal{M}^*$ for which the relation

$$\sum_{n=1}^{\infty} \frac{1}{n} |g(n+k) - f(n)| < \infty \tag{3.1}$$

with some fixed $k \in \mathbb{N}$ holds. Namely I proved the following assertions.

Theorem 12 *If $f, g \in \mathcal{M}$ and* (3.1) *holds with $k = 1$, then either*

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n} < \infty, \quad \sum_{n=1}^{\infty} \frac{|g(n)|}{n} < \infty, \tag{3.2}$$

or

$$f(n) = g(n) = n^{\sigma + i\tau}, \quad \sigma, \tau \in \mathbb{R}, \quad 0 \le \sigma < 1. \tag{3.3}$$

Theorem 13 *Let $f, g \in \mathcal{M}^*$ and $k \ge 2$ be fixed. Assume that* (3.1) *holds, furthermore that $f(n) = g(n) = 0$ if $(n,k) > 1$ and $f(n) \ne 0, g(n) \ne 0$ if $(n,k) = 1$. Then either* (3.2) *is satisfied or there exist $F, G \in \mathcal{M}^*$ and $s \in \mathbb{C}$ with $\Re s < 1$, such that $f(n) = n^s F(n), g(n) = n^s G(n)$, and*

$$G(n+k) = F(n) \quad (n \in \mathbb{N}) \tag{3.4}$$

holds.

In [18, IV.] I determined all the solutions of (3.4) for completely multiplicative pairs of F, G and in [19] even for $F, G \in \mathcal{M}$ under the additional condition that $F(n) \neq 0$ if $(n, k) = 1$. The above assertions are not obvious even in the case $g = f$.

An immediate consequence of Theorem 12 is that $\sum_{n=1}^{\infty} \frac{1}{n}|\lambda(n+1) - \lambda(n)| = \infty$, where λ is the Liouville function. This shows that the size of the integers n for which $\lambda(n) \neq \lambda(n+1)$ is not too small.

Recently in a joint paper with B.M. Phong [20] we proved

Theorem 14 *Let $k \in \mathbb{N}$ be fixed. Assume that $F, G \in \mathcal{M}$ and (3.4) is satisfied. Then either*

$$S_F := \{n | F(n) \neq 0\} \quad \text{and} \quad S_G := \{n | G(n) \neq 0\}$$

are finite sets, or $F(n) \neq 0$ for every n coprime to k.

A special case was treated earlier in [21].

In [22] I formulated the following

Conjecture 1 If $f \in \mathcal{M}$ and

$$\frac{1}{x} \sum_{n \leq x} |\Delta f(n)| \longrightarrow 0, \tag{3.5}$$

then either

$$\frac{1}{x} \sum_{n \leq x} |f(n)| \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty, \tag{3.6}$$

or $f(n) = n^s$, $\Re s < 1$.

Towards this conjecture, a few partial results are known.

First, assuming that (3.6) does not hold, from (3.5) one can deduce that $f \in \mathcal{M}^*$. This assertion was explicitly proved by Mauclaire and Murata [23] for functions f of modulus 1, but their method can be applied to the general case.

The second observation is that either $|f(n)| \geq 1$ for every n, or (3.6) holds. Indeed, let $|f(q)| = \varrho < 1$, $S(x) := \sum_{n \leq x} |f(n)|$. Then

$$S(x) \leq \sum_{m=1}^{[x/q]+1} \sum_{j=0}^{q-1} |f(mq+j)| \leq \sum_{m=1}^{[x/q]+1} q|f(mq)| + \sum_{m} \sum_{j} |f(mq+j) - f(mq)|.$$

According to (3.5), the second sum on the right hand side is smaller than ε, ($\varepsilon > 0$ arbitrary), if x is large enough, the first sum is $q\varrho S([\frac{x}{q}] + 1)$, consequently,

$$\frac{S(x)}{x} \leq \varepsilon + \varrho \frac{S([\frac{x}{q}] + 1)}{x/q},$$

whence $S(x)/x \to 0$ immediately follows.

Moreover arguing similarly, one can deduce that if (3.6) does not hold, then $|f(n)| = n^\sigma$ with a constant σ, $0 \le \sigma < 1$. Let $t(n) := f(n)n^{-\sigma}$, and assume that $\sigma > 0$. Since $t(n+1) - t(n) = f(n+1)((n+1)^{-\sigma} - n^{-\sigma}) + (\Delta f(n))n^{-\sigma}$, therefore

$$\sum_{n\le x} \frac{|\Delta t(n)|}{n} \ll \sum_{n\le x} \frac{1}{n^2} + \sum_{n\le x} \frac{|\Delta f(n)|}{n^{\sigma+1}}.$$

The right hand side is clearly convergent, therefore Theorem 12 can be applied, whence we obtain that $t(n) = n^{i\tau}$, $\tau \in \mathbb{R}$, i.e. $f(n) = n^s$, $0 < \Re s < 1$.

The case, when $f(n)$ is of modulus 1 seems to be very hard. Hildebrand [23] proved

Theorem 15 *There exists a positive constant c with the following property. If $g \in \mathcal{M}^*$, $|g(n)| = 1$ for $n \in \mathbb{N}$ and for every $p \in \mathcal{P}$, $|g(p) - 1| \le c$, then either $g(n) = 1$ identically, or*

$$\liminf \frac{1}{x} \sum_{n\le x} |\Delta g(n)| > 0. \tag{3.7}$$

By using the ideas of Hildebrand and some of mine, I obtained [18, VI.]

Theorem 16 *Let $g \in \mathcal{M}^*$, $|g(n)| = 1$ for $n \in \mathbb{N}$. There exist positive constants $\beta < 1$ and δ such that*

$$\limsup \sum_{x^\beta < p < x} \frac{|g(p) - 1|}{p} < \delta \tag{3.8}$$

and

$$\liminf \frac{1}{x} \sum_{x/2\le n\le x} |\Delta g(n)| = 0 \tag{3.9}$$

imply that $g(n) = 1$.

Let $\varrho : [1, \infty) \to [1, \infty)$ be a slowly varying function, i.e. such that

$$\lim_{x\to\infty} \max_{y\in[x/2,x]} \left| \frac{\varrho(y)}{\varrho(x)} - 1 \right| = 0.$$

Let Ω denote the set of all arithmetical functions having complex values. $f \in \Omega$ is considered as an infinite dimensional vector, the n'th coordinate of which is $f(n)$. Let $\alpha \ge 1$ be a constant and $\Omega_{\alpha,\varrho}$ be the subspace of Ω which consists of those $x \in \Omega$ for which

$$\sup_{y\ge1} \frac{1}{y\varrho(y)^\alpha} \sum_{n\le y} |x_n|^\alpha$$

is finite.

Let $\mathcal{L}_{\alpha,\varrho} = \mathcal{M} \cap \Omega_{\alpha,\varrho}$, $\mathcal{L}^*_{\alpha,\varrho} = \mathcal{M}^* \cap \Omega_{\alpha,\varrho}$.

In a joint paper with Indlekofer [24] we proved

Theorem 17 *If* $f \in \mathcal{M}$, $P \in C[z]$, $P \neq 0$, $k = \deg P$, *and*

$$P(E)f \in \Omega_{\alpha,\varrho}$$

then either $f \in \mathcal{L}_{\alpha,\varrho}$, *or* $f(n) = n^s u(n)$, *where* $0 \le \Re s \le k$ *and*

$$P(E)u = 0$$

The next assertion was proposed by myself as a conjecture and proved by Wirsing in 1984.

Theorem 18 *If* $f \in \mathcal{M}$, $\Delta f(n) \to 0$ *as* $n \to \infty$, *then* $f(n) \to 0$ *as* $n \to \infty$ *or* $f(n) = n^s$, $0 \le \Re s < 1$.

This theorem has been proved some years later independently by Tang and Shao. The joint paper of Wirsing, Tang and Shao [25] contains two different proofs.

Wirsing's theorem can be formulated in the following way: If $F \in \mathcal{A}$ and $\|\Delta F(n)\| \to 0$, then with some suitable constant $\lambda \in \mathbb{R}$ we have that $F(n) - \lambda \log n$ is integer for every $n \in \mathbb{N}$.

In other words, if T is the group of the reals mod 1, and $F \in \mathcal{A}_T$, $\Delta F(n) \to 0$, then F is a restriction of a continuous homomorphism from $\mathbb{R}_\times$ to T.

B.M. Phong proved the following generalization of Wirsing's theorem.

Theorem 19 *Let* A, B *be positive integers and let* D *be a real constant. If* $h \in \mathcal{A}_T^*$ *and*

$$h(An + B) - h(n) - D \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

then h *is the restriction of a continuous homomorphism:* $\mathbb{R}_\times \to T$.

For $A = 1$ this assertion was generalised by Tang [29]:

Theorem 20 *Let* B *be a fixed positive integer,* f *a multiplicative function defined on the set of the integers* n *coprime to* B, *such that* $|f(n)| = 1$ *and* $f(n + B) - f(n) \to 0$, $n \to \infty$. *Then there must be a* $\tau \in \mathbb{R}$ *such that* $f(n) = n^{i\tau}\chi_B(n)$, *where* $\chi_B(n)$ *is a Dirichlet-character mod* B.

By using this assertion one can completely characterize all those multiplicative functions f of modulus 1, for which $P(E)f(n) \to 0$, $(n \to \infty)$ holds. (For this see [18, I])

In a joint paper with N.L. Bassily [28] we proved

Theorem 21 *If* $f, g \in \mathcal{M}$ *and* $g(2n+1) - Cf(n) \to 0$ *with some nonzero constant* C, *then either* $f(n) \to 0$ *as* $n \to \infty$, *or* $C = f(2)$, $f(n) = n^s$, $0 \le \Re s < 1$, *and* $g(n) = f(n)$ *for every odd* n.

The complete description of those $f, g \in \mathcal{M}$ for which $g(An + B) - Cf(an + b) \to 0$ $(n \to \infty)$ is not given yet.

4 On Additive Functions mod 1

T is considered here as the additive group $\mathbb{R}/\mathbb{Z}$. We say that $F \in \mathcal{A}_T$ is of finite support if $F(p^\alpha) = 0$ holds for every large prime p, and every $\alpha \in \mathbb{N}$. For $F_\nu \in \mathcal{A}_T(\nu = 0, 1, \ldots, k-1)$ let

$$L_n(F_0, \ldots, F_{k-1}) := F_0(n) + \cdots + F_{k-1}(n+k-1). \tag{4.1}$$

Conjecture 2 Let $\mathcal{L}_0^{(k)}$ be the space of those k-tuples $(F_0, \ldots, F_{k-1})$ of $F_\nu \in \mathcal{A}_T$ for which

$$L_n(F_0, \ldots, F_{k-1}) = 0 \quad (n \in \mathbb{N}) \tag{4.2}$$

holds. Then each F_j is of finite support, and $\mathcal{L}_0^{(k)}$ is a finite dimensional $\mathbb{Z}$ module. Let $G_j(n) = \tau_j \log n \pmod 1$, $\tau_0 + \cdots + \tau_{k-1} = 0$. Then $L_n(G_0, G_1, \ldots, G_{k-1}) \to 0$ as $n \to \infty$.

Conjecture 3 If $F_\nu \in \mathcal{A}_T(\nu = 0, \ldots, k-1)$, and

$$L_n(F_0, \ldots, F_{k-1}) \longrightarrow 0 \quad (n \longrightarrow \infty),$$

then there exist suitable real numbers $\tau_0, \ldots, \tau_{k-1}$ such that $\tau_0 + \cdots + \tau_{k-1} = 0$, and for $H_j(n) := F_j(n) - \tau_j \log n$ we have

$$L_n(H_0, \ldots, H_{k-1}) = 0 \quad (n = 1, 2, \ldots).$$

Remarks:

1. Conjecture 3 for $k = 1$ can be deduced easily from Wirsing's theorem.
2. Conjecture 2 was proved for $k = 3$ under the more strict condition that $F_\nu \in \mathcal{A}_T^*$ in [30]. We obtained that (4.2) implies that $F_\nu = 0$ $(\nu = 0, 1, 2)$ identically.
3. Conjecture 2 for $k = 3$ was proved completely by R. Styer [31].
4. M. Wijsmuller treated similar problems for additive functions defined on the set of Gaussian integers taking values from T. See [32], [33].

Let $P(n)$ be the largest and $p(n)$ the smallest prime divisor of n.

Conjecture 4 For every integer $k(\geq 1)$ there exists a constant c_k such that for every prime p greater than c_k,

$$\min_{\substack{1 \leq j \\ P(j) < p}} \max_{\substack{i \in [-k, k] \\ i \neq 0}} P(jp + l) < p \tag{4.3}$$

holds.

We are unable to prove it even for $k = 2$.

Proposition 1 *Let $\mathcal{L}_0^{*(l)}$ be the space of those l-tuples $(F_0, \ldots, F_{l-1})$ of $F_\nu \in \mathcal{A}_T^*$ for which $L_n(F_0, \ldots, F_{l-1}) = 0$ $(n \in \mathbb{N})$. Assume that Conjecture 4 is true for $k = l$. Then $\mathcal{L}_0^{*(l)}$ is a finite dimensional space.*

Proof: Let $(\hat{F}_0, \ldots, \hat{F}_{l-1})$ be such an element of $\mathcal{L}_0^{*(l)}$ for which $\hat{F}_j(q) = 0$ for every $q \le \max(c_l, l)$ and $j = 0, \ldots, l-1$. We shall prove that $\hat{F}_j(n) = 0$ for every $n \in \mathbb{N}$, $j = 0, \ldots, l-1$. Assume the contrary, and let M be the smallest integer for which $\hat{F}_t(M) \neq 0$ for some $t \in \{0, \ldots, l-1\}$. Then M should be a prime. Since

$$L_{jM-t}(\hat{F}_0, \ldots, \hat{F}_{l-1}) = \sum_{i=0}^{l} \hat{F}_i(jM - T + i) = 0,$$

from (4.3), by choosing that j for which (4.3) is attained (with $M = p$), we obtain that $\hat{F}_t(M) = 0$.

Hence it follows that the initial values $F_j(q)$, $j = 0, \ldots, l-1$; $q \le \max(c_l, l)$ completely determine the functions F_j, if they are correlated according to (4.2).

The proof is complete. □

Let K be the closure of the set $\{L_n(F_0, \ldots, F_{k-1}) | n \in \mathbb{N}\}$.

Conjecture 5 If $F_0, \ldots, F_{k-1} \in \mathcal{A}_T^*$ and K contains an element of infinite order, then $K = T$.

This conjecture is obvious if $k = 1$, and it seems to be hard for $k \ge 2$. Recently, in our joint papers with M.V. Subbarao [34], [35] we obtained some partial results. This will be explained in the remaining part of this section.

Let $E_k = \{u/k | u = 0, 1, \ldots, k-1\}$, i.e. the group of those elements $\alpha \in T$ for which $k\alpha = 0$. A special case of Conjecture 5 would be

Conjecture 6 Let $f \in \mathcal{A}_T^*$, and $\mathcal{H} = \{\alpha_1, \ldots, \alpha_k\}$ be the set of the limit points of the sequence $f(n+1) - f(n) (n \in \mathbb{N})$. Then $\mathcal{H} = E_k$, and there exists a real number τ such that $f(n) = \tau \log n + U(n) \pmod 1$, $U(\mathbb{N}) = E_k$, and for every $\omega \in E_k$ there exists a subsequence n_ν of integers such that $U(n_\nu + 1) - U(n_\nu) = \omega$.

We proved

Theorem 22

1) *Conjecture 6 is true for* $k = 1, 2, 3$.
2) *Let* $k = 4$, *and assume that the conditions of Conjecture 6 are satisfied. Then there is a* $\tau \in \mathbb{R}$ *such that* $f(n) = \tau \log n + U(n) \pmod 1$ *and either (A) or (B) hold:*
   - (A) $\mathcal{H} = E_4$, $U(\mathbb{N}) \subseteq E_4$
   - (B) $\mathcal{H}$ *consists of four distinct elements of* E_5, *i.e.* $\mathcal{H} = \{\kappa^{l_1}, \kappa^{l_2}, \kappa^{l_3}, \kappa^{l_4}\}$, *where* κ *is any nonzero element of* E_5, *moreover* $U(\mathbb{N}) \subseteq E_5$ *and* $U(n+1) - U(n) \in \mathcal{H}$ *for every large* n.

Remark: We think that case (B) cannot hold, which would follow if we could prove that $U(\mathbb{N}) = E_5$ implies that for every $\alpha \in E_5$, $U(n+1) - U(n) = \alpha$ occurs infinitely often.

5 Characterization of Continuous Homomorphisms as Elements of $\mathcal{A}_G$ for Compact Groups

We investigated this topic in a series of papers written jointly with Z. Daróczy [36–41].

Assume in this section that G is a metrically compact Abelian group supplied with some translation invariant metric ϱ. An infinite sequence $\{x_n\}_{n=1}^{\infty}$ in G is said to belong to $\mathcal{E}_D$, if for every convergent subsequence $x_{n_1}, x_{n_2}, \ldots$ the "shifted subsequence" $x_{n_1+1}, x_{n_2+1}, \ldots$ is convergent, too. Let $\mathcal{E}_\Delta$ be the set of those sequences $\{x_n\}_{n=1}^{\infty}$ for which $\Delta x_n = x_{n+1} - x_n \to 0$ $(n \to \infty)$ holds. Then $\mathcal{E}_\Delta \subseteq \mathcal{E}_D$. We say that $f \in \mathcal{A}_G^*$ belongs to $\mathcal{A}_G^*(\Delta)$ (resp. $\mathcal{A}_G^*(D)$) if the sequence $\{f(n)\}_{n=1}^{\infty}$ belongs to $\mathcal{E}_\Delta$ (resp. $\mathcal{E}_D$).

We proved the following assertions.

(1) $\mathcal{A}_G^*(\Delta) = \mathcal{A}_G^*(D)$.

(2) If $f \in \mathcal{A}_G^*(D)$, then there exists a continuous homomorphism $\Phi : \mathbb{R}_\times \to G$ such that $f(n) = \Phi(n) (n \in \mathbb{N})$.
The proof of (2) was based upon the theorem of Wirsing (Theorem 18).
The set of all limit points of $\{f(n)\}_{n=1}^{\infty}$ form a compact subgroup in G which is denoted by S_f.

(3) $f \in \mathcal{A}_G^*(D)$ if and only if there exists a continuous function $H : S_f \to S_f$ such that $f(n+1) - H(f(n)) \to 0$ as $n \to \infty$.

(4) In [41] we characterized those $f \in \mathcal{A}_G^*$ for which with some continuous function $F : S_f \to S_f$ the relation $f(2n-1) - F(f(n)) \to 0$ $(n \to \infty)$ holds. For $G = T$ we obtained that either $f(n) = 0$ for every odd n, or there exists a nonzero $\lambda \in \mathbb{R}$ such that $f(n) = \lambda \log n (\text{mod } 1)$ for every $n \in \mathbb{N}$.

(5) In [44] we solved the following problem. Let G_1, G_2 be metrically compact Abelian groups with some translation invariant metrics. Let $f \in \mathcal{A}_{G_1}^*, g \in \mathcal{A}_{G_2}^*$, and assume that with some continuous function $F : S_f \to S_g$ the relation $g(n-1) - F(f(n)) \to 0$ $(n \to \infty)$ holds. E.g. for $G_1 = T$ we proved: Under the above conditions, either $g(n) = 0$ identically, or there exist $\tau \in \mathbb{R}, M \in \mathbb{N}, u \in \mathcal{A}_{E_M}$ such that $f(n) = \frac{\tau}{M} \log n + u(n) \bmod 1$. Let $\lambda(n) := Mf(n) (n \in \mathbb{N})$. Then the correspondence $\lambda(n) \leftrightarrow g(n) (n \in \mathbb{N})$ generates a topological isomorphism between S_λ and S_g. The converse assertion is also true.

(6) Further interesting results were obtained by Phong [42], [43].

(7) The main problem we are interested in is the following one:
Let $f_j \in \mathcal{A}_{G_j} (j = 0, \ldots, k-1)$, G_j be compact groups, $e_n := \{f_0(n), f_1(n+1), \ldots, f_{k-1}(n+k-1)\}$. Then $e_n \in S_{f_0} \times \cdots \times S_{f_{k-1}} (=: U)$. What can we say about the functions f_j if the set of limit points is not everywhere dense in U? We shall formulate our guesses only for special cases.

Conjecture 7 Let $f \in \mathcal{A}_T^*, S_f = T, e_n = (f(n), \ldots, f(n+k-1))$. Then, either $f(n) = \lambda \log n$ (mod 1) with some $\lambda \in \mathbb{R}$, or $\{e_n | n \in \mathbb{N}\}$ is dense in $T_k = T \times \cdots \times T$.

Conjecture 8 Let $f, g \in \mathcal{A}_T^*, S_f = S_g = T, e_n := (f(n), g(n+1))$. If e_n is not everywhere dense in $T_2 = T \times T$, then f and g are rationally dependent continuous homomorphisms, i.e. there exist $\lambda \in \mathbb{R}, s \in \mathbb{Q}$ such that $g(n) \equiv sf(n) (\text{mod } 1)$, $f(n) = \lambda \log n$ (mod 1).

Mauclaire proved in [45] that if G is an arbitrary locally compact group and $f \in \mathcal{A}_G$ satisfies $\Delta f(n) \to 0$ $(n \to \infty)$ then f is the restriction of a continuous homomorphism $\varphi : \mathbb{R}_\times \to G$. Ruzsa and Tijdeman proved [46] that it cannot be generalized for all groups.

6 Sets of Uniqueness for Completely Additive Functions

Definition: We say that $E \subseteq \mathbb{N}$ is a set of uniqueness for the functions belonging to $\mathcal{A}^*$ if $f \in \mathcal{A}^*$, $f(E) = 0$ implies that $f(\mathbb{N}) = 0$.

I introduced this notion in [47], and in [48] it was proved that if to the sequence of "prime + one"s we adjoin a finite set of integers then we obtain a set of uniqueness. My guess that the set of shifted primes itself is a set of uniqueness, was proved by Elliott [49].

It was proved by Wolke [49], and Dress and Volkman [50], that in order for a set E to be such a set of uniqueness, it is necessary and sufficient that every positive integer n has a multiplicative representation:

$$n^k = \prod_{i=1}^{k} a_{j_i}^{\varepsilon_i} \quad a_{j_i} \in E, \quad \varepsilon_i = \pm 1.$$

The h, k may vary with n. They used vector spaces over the field of rational numbers.

In [52] Elliott proved my further conjecture, namely that if $f \in \mathcal{A}^*$, $M(x) = \max_{n \le x} |f(n)|$, $E(x) = \max_{p \le x} |f(p+1)|$, then

$$M(x) \le AE(x^B) \quad x \ge 2 \tag{6.1}$$

holds with suitable numerical constants A, B. For the wider class $f \in \mathcal{A}$ he got a weaker result, namely that

$$M(x) \le AE(x^B) + AM((\log x)^C)$$

for some $C > 0$.

Wirsing extended (6.1) for $f \in \mathcal{A}$ [53]. He proved that every $n \in \mathbb{N}$ has a representation

$$n^h = \prod_{i=1}^{k} (p_i + 1)^{\varepsilon_i}$$

where h and k are bounded, $\varepsilon_i = \pm 1$, and the primes p_i lie in an interval $n < p_i \le n^B$. In particular, Wirsing's result showed that for the multiplicative group K generated by the "prime plus one"s $\mathbb{Q}_\times / K$ has bounded order.

Another interesting consequence of Wirsing's result is that $f \in \mathcal{A}$, $f(p+1) \to 0$ $(p \in \mathcal{P})$ implies that $f(n) = 0$.

My motivation with the investigation of the set of shifted primes was the following. In 1968 I proved [54] that $f \in \mathcal{A}$ has a limit distribution on the set of shifted primes if the three series

$$\sum_{|f(p)| \le 1} \frac{f(p)}{p}, \quad \sum_{|f(p)| \le 1} \frac{f^2(p)}{p}, \quad \sum_{|f(p)| \ge 1} \frac{1}{p} \tag{6.2}$$

are convergent. But the question of the necessity of these conditions remained open.

The necessity of the convergence of the series was proved by additional assumptions: a) if $f(p) \geq 0$, by Elliott [55]; b) if $f(p) = \mathcal{O}(1)$, by Kátai [56]. Finally it was proved without any other conditions by Hildebrand [58] in 1988. From his result it follows that, if $f \in \mathcal{A}$ satisfies

$$\frac{1}{\pi(x)}\#\{p \leq x : |f(p+1)| \geq \varepsilon\} \longrightarrow 0 \quad (x \longrightarrow \infty)$$

for every $\varepsilon > 0$, then $f(n) = 0$ identically.

The notion of sets of uniqueness can be extended into group valued arithmetical functions.

Definition 2 Let G be an arbitrary Abelian group. We say that $E \subseteq \mathbb{N}$ is a set of uniqueness for the class of functions in $\mathcal{A}_G^*$ if $f \in \mathcal{A}_G^*$, $f(E) = 0$ implies that $f(\mathbb{N}) = 0$.

For $G = T$ the following assertion has been proved by Meyer [58], Indlekofer [59], Dress and Volkman [51], see also Elliott [60]:

In order that E would be a set of uniqueness for the class $\mathcal{A}_T^*$ it is necessary and sufficient that every positive integer n has a representation

$$n = \prod_{j=1}^{s} a_j^{d_j}$$

with some integers d_j, positive, negative or zero.

Probably, the set of "prime plus one"s is a set of uniqueness for $\mathcal{A}_T^*$ but it does not seem to be easy. Presently it is not disproved even that $\frac{1}{2}\Omega(p+1) \equiv 0 \pmod 1$ for every large p.

In my paper [15] implicitly it was proved that there is a constant L such that every integer n has a representation

$$n = A\prod_{i=1}^{k}(p_i + 1)^{\varepsilon_i}, \quad \varepsilon_i = \pm 1,$$

where A is such a rational number in the reduced form of which all prime factors are less than L. The constant L was implicit, since I used the Bombieri-Vinogradov theorem. Later Elliott [61] proved that $L = 10^{387}$ is appropriate.

This bound is extremely large for computation. If we could reduce it to 10^{12}, say, then with a massive computer calculation perhaps we could prove that $K = \mathbb{Q}_\times$.

Recently Elliott [62] proved that the factor group $\mathbb{Q}_\times / K$ is either trivial or is of order 2, or 3.

Schinzel and Sierpinski in 1958 stated the conjecture [62], that every positive rational has infinitely many representations of the form $(p+1)(q+1)^{-1}$ with $p, q \in \mathcal{P}$. From this $K = \mathbb{Q}_\times$ would immediately follow.

By using the method of Chen [63] one can prove that every natural number n has infinitely many representations of the form $(P_2+1)(Q_2+1)^{-1}$, where P_2, Q_2 run over the integers the number of prime factors of which is at most 2. Consequently the multiplicative group K_1 generated by set $P_2 + 1$, where P_2 runs over the integers having at most two prime factors equals to $\mathbb{Q}$.

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